THE CONSTRUCTION OF THE WINDOW[X] FUNCTION (used in time - series and signal processing)

Start by reading the output of this notebook.

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(using *Mathematica* 5.1)
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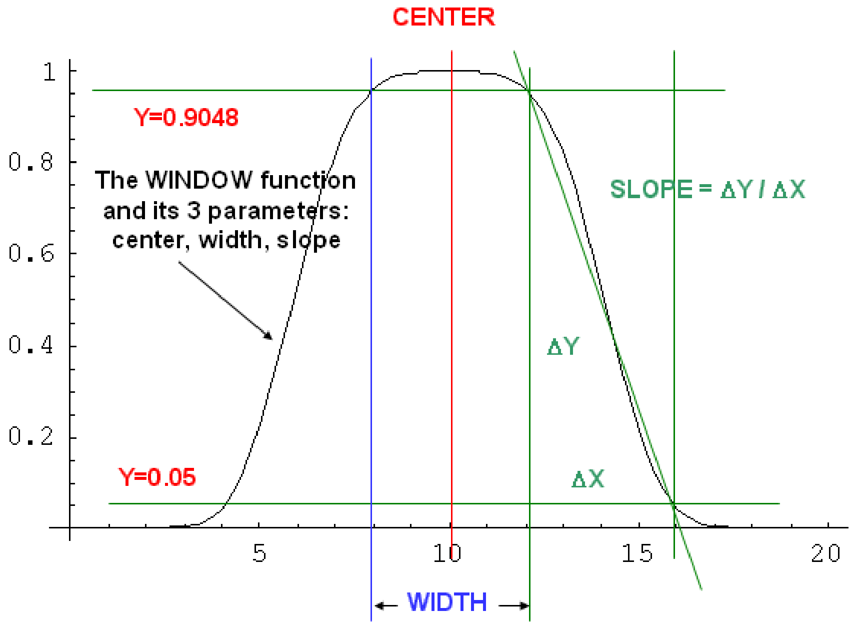



```
Clear[f, x, c, s, w, center, width, slope];

(* We start with function f[x]: *)

    f[x_] := Exp[-((x - c)/s)^w];

(* Note that when w=2, f is the Gauss (or Bell) function *)

    c = 10; s = 4.1; w = 10;
    Print["The Bell Curve (blue) and Window[x] (red) : ", f[x]];
    Plot[{f[x], Exp[-((x - c)/s)^2]}, {x, 0, 20}, AxesOrigin → {0, 0},
    PlotRange → All, PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}];

(*
Now we change the variables from {c,w,s} to {center,width,slope},
    according to the figure shown above.
```

The value of x at the center of the curve is c = center.
When f[x]=0.9048, x = x1 > c and width = 2 * (x1 - c):

Solve[f==0.9048,x] (* to find X1 *);

$x1=c+\frac{s}{10^{\frac{1}{w}}}$

width = $\frac{2\ s}{10^{\frac{1}{w}}}$; isolating s on the left:

s = $\frac{\text{width}}{2}\ 10^{\frac{1}{w}}$;

Now verify this result.
*)

Clear[w, c];
c = center;

s = $\left(\frac{\text{width}}{2}\right)\left(10^{\frac{1}{w}}\right)$;

center = 10; width = 4; w = 4;
Print["Center = ", center];
Print["Plot f[x] for width = ", width]
Print["Function f[x] being plotted: ", f[x]];
Plot[f[x], {x, 0, 20}, AxesOrigin → {0, 0}, PlotRange → All]

(*
Now we search for the value of the slope = $\frac{\Delta y}{\Delta x}$; (see figure above);

$x1=c+\frac{s}{10^{\frac{1}{w}}}$; $f=e^{-\left(\frac{x-c}{s}\right)^{w}}$; (* already found *)

Solve[f==0.05,x] (* find x2 *);

$x2=c+s*3^{\left(\frac{1}{w}\right)}$;

$\Delta x=x2-x1$; $\Delta y=0.9$;

slope= $\frac{\Delta y}{\Delta x}$;

slope= $\frac{0.9}{\left(3^{\frac{1}{w}}-10^{-\frac{1}{w}}\right)\ s}$; isolating s on the left:

s= $\frac{0.9}{\left(3^{\frac{1}{w}}-10^{-\frac{1}{w}}\right)\ \text{slope}}$;

But

s = $\frac{\text{width}}{2}\ 10^{\frac{1}{w}}$; (* See above *)

Solving this system of 2 eqs & 2 unknowns:

w = $\frac{\text{Log}[30]}{\text{Log}\left[1+\frac{1.8}{\text{slope}\ *\ \text{width}}\right]}$

Once defined the values of {w,s,c}, we get a pre-definition of Window[x]:

k = 1 + $\frac{1.8}{\text{slope}\ *\ \text{width}}$;

preWindow[x] = Exp$\left[-\left(\frac{2\ k^{-\frac{\text{Log}[10]}{\text{Log}[30]}}\ (x-c)}{\text{width}}\right)^{\frac{\text{Log}[30]}{\text{Log}[k]}}\right]$;

Because Window[x] is an even function, we must use Abs[x-c] instead of (x-c),
or (which is the same) making $(x-c)\to\left((x-c)^{2}\right)^{\frac{1}{2}}$, a continuous function.

With above considerations, Window[x] finally becomes:

Window[x] = Exp$\left[-\left(k^{-\text{Log}[10]}\left(\left(\frac{x-\text{center}}{\text{width}/2}\right)^{2}\right)^{\frac{\text{Log}[30]}{2}}\right)^{\frac{1}{\text{Log}[k]}}\right]$

Note that Window[x] is a function of the type u^{x^z}.

To avoid division by zero,

Log[k] > 0, k > 1, 0 < width < ∞, 0 < slope < ∞;

If Abs[x-c] is too big, underflows may result during computation.
To avoid this, we define the range of Window[x]:

0 ≤ Window[x] < min

where

min = $2.225\ 10^{-308}$ for a double-precision floating-number in C++ (float.h);
min = $5.208\ 10^{-646456888}$ in Mathematica (calling $MinNumber);

or

min = Exp[-708.4]   (C++)
min = Exp[$-1.48\ 10^{9}$]   (Mathematica)

To find the maximum of Abs[x-c], we equate

Window[xuf] == min   (* xuf → min x that causes underflow *)

to obtain the condition to avoid underflow:

$$\text{Abs[xuf-c]} < \left(\frac{\text{width}}{2}\right) * k^{\left(\frac{\text{Log[maxExpo] + Log[10]}}{\text{Log[30]}}\right)}$$

where

maxExpo = 708.4   (for C++)
maxExpo = $1.48\ 10^{9}$ (for Mathematica)

For C++

$$\text{Abs[x-c]} < \frac{\text{width}}{2}\ k^{2.6066}$$

For Mathematica

$$\text{Abs[x-c]} < \frac{\text{width}}{2}\ k^{6.88}$$

To reduce the time used to compute Window[x], we may observe that

if x >> c, Window[x] → zero.

Then, if

Window[xfar > c] == 0.001

the value of xfar is

$$\text{Abs[xfar-c]} = \frac{\text{width}}{2}\ k^{1.245}$$

Also, when window[x05]==0.05 → (x05-center) = $\frac{\text{width}}{2}$ k

We note that:

$$\text{Abs[x05-center]} = \frac{\text{width}}{2}\ k < \text{Abs[xfar-c]} = \frac{\text{width}}{2}\ k^{1.245} <$$
$$< \text{Abs[xuf1-c]} = \frac{\text{width}}{2}\ k^{2.6066} \ll \text{Abs[xuf2-c]} = \frac{\text{width}}{2}\ k^{6.88}$$

In words,

(1) xfar is less then the underflow limit xuf
(2) xfar is near x05

We may resume the above considerations with the command:

```
If[ Abs[x-c] > (width/2) k^1.245,
    Return[0],
    Return[Window[x];
]
```

```
A complete definition of Windows[x] (in red) is shown in the next Module[].
    *)

(* Mathematical expression of window[x]:

    window[x_,center_,width_,slope_]:= Exp[-( ((x-center)/(width/2))^2)^(Log[30]/2) / k^Log[10] )^(1/Log[k]) ];

*)

    Clear[slope, width, center, k, window];
    k := 1 + 1.8/(slope * width);

    window[x_, center_, width_, slope_] :=
        Module[{},
            If[ width ≤ 0, Print["Width must be > 0: ", width]; Return["ERROR"]];
            If[ slope ≤ 0, Print["Slope must be > 0: ", slope]; Return["ERROR"]];
            If[Abs[x - center] > (width/2) k^1.245, Return[0]];

            Return[ Exp[-( k^-Log[10] ((x - center)/(width / 2))^2)^(Log[30]/2) )^(1/Log[k]) ]]

            ];

    "center=0;width=6;slope=2;"
    center = 0; width = 6; slope = 2;

    Print["Plotting window[x] for slope = ", slope];
    Print[window[x, center, width, slope]];
    Plot[window[x, center, width, slope],
  {x, -10, 10}, AxesOrigin → {0, 0}, PlotRange → All];

     (* Now plot window[x] when center, width and slope change *)

Print[];
Print["Window[x] when its center is changed (below)"];
Clear[x, center];
"width=10; slope=0.5;"
width = 10; slope = 0.5;
Print[window[x, center, width, slope] // TraditionalForm];
Plot3D[window[x, center, width, slope], {x, -40, 40}, {center, -10, 10},
  PlotRange → All, AxesLabel → {"X value", "Center Point", "Window[x]"},
  Mesh → False, FaceGrids → All, PlotPoints → 100];

Print[];
Clear[x, width];
Print["Window[x] when its width is changed (below)"];
Clear[width];
"center=0;slope=0.06;"
center = 0; slope = 0.06;
Print[window[x, center, width, slope] // TraditionalForm];
Plot3D[window[x, center, width, slope], {x, -40, 40},
```

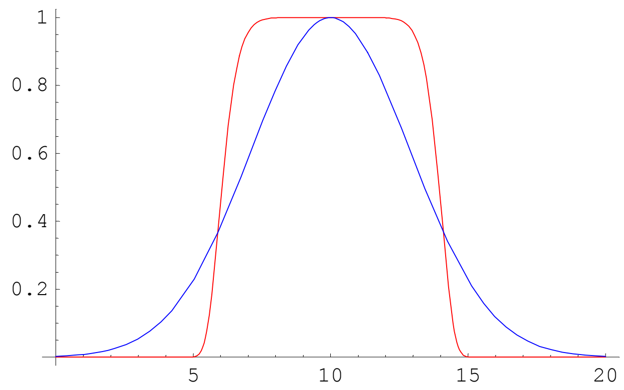
```
  {width, 2, 40}, PlotRange → All, AxesLabel → {"X value", "Width", "Window[x]"},
  Mesh → False, FaceGrids → All, PlotPoints → 100];

Print[];
Clear[x, slope];
Print["Window[x] when its slope is changed (below)"]
Clear[slope];
"center=0;width=50;"
center = 0; width = 50;
Print[window[x, center = 0, width, slope] // TraditionalForm];
Plot3D[window[x, center = 0, width, slope], {x, -600, 600},
  {slope, 0.001, .03}, PlotRange → All, AxesLabel → {"X value", "Slope", "Window[x]"},
  Mesh → False, FaceGrids → All, (*ColorOutput→GrayLevel,*) PlotPoints → 200];

(* END OF JOB *)
```

Null

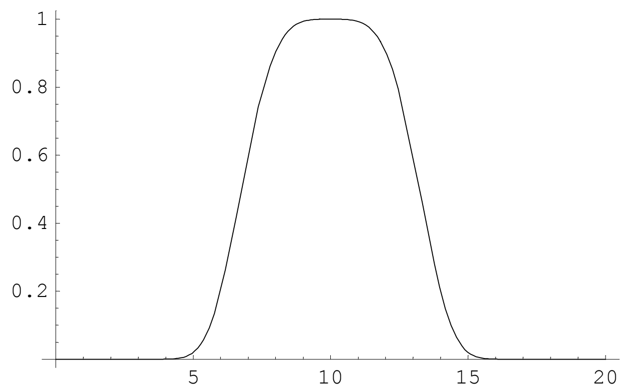
The Bell Curve (blue) and Window[x] (red) : $e^{-7.45009\times10^{-7}\,(-10+x)^{10}}$



Center = 10

Plot f[x] for width = 4

Function f[x] being plotted: $e^{-\frac{1}{160}\,(-10+x)^{4}}$

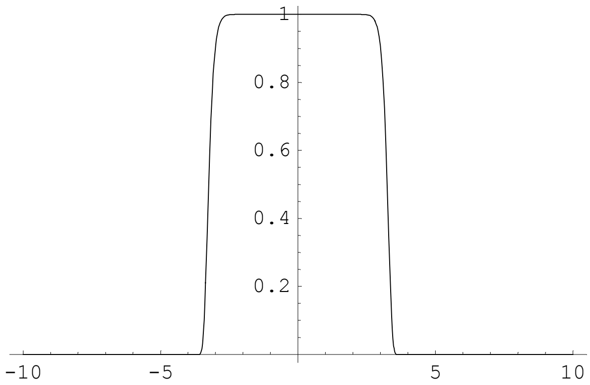


- Graphics -

center=0;width=6;slope=2;

Plotting window[x] for slope = 2

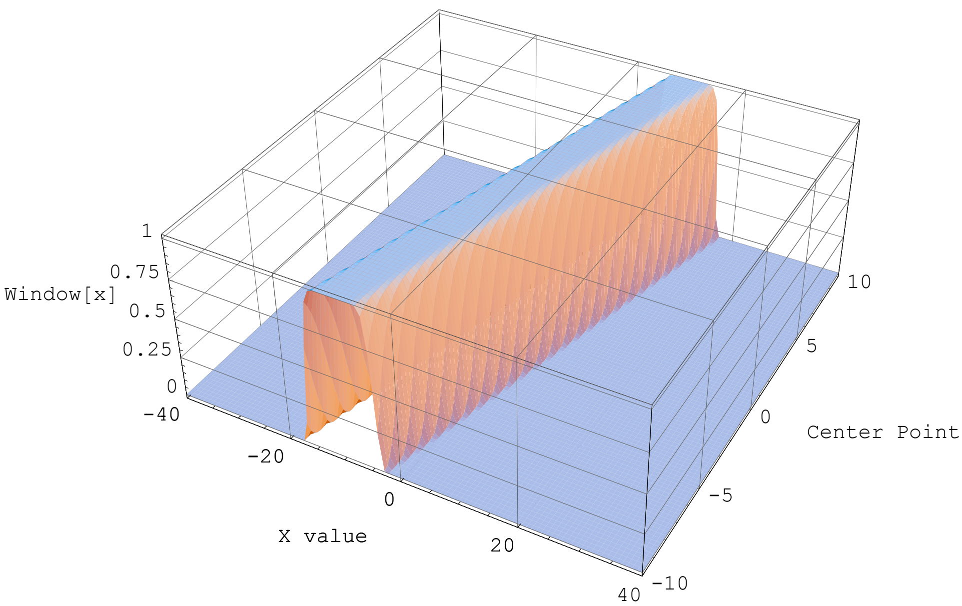
$$e^{-2.44875\times10^{-13}\left((x^2)^{\frac{\text{Log}[30]}{2}}\right)^{7.15502}}$$



Window[x] when its center is changed (below)

width=10; slope=0.5;

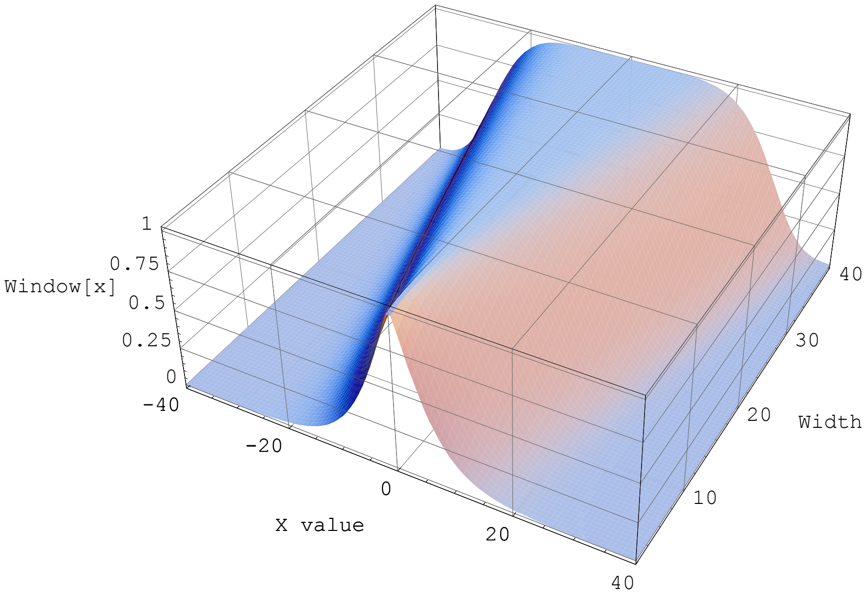
$$e^{-1.85543\times10^{-9}\left(((x-\text{center})^2)^{\frac{\log(30)}{2}}\right)^{3.25219}}$$



Window[x] when its width is changed (below)

center=0;slope=0.06;

$$e^{-2^{\frac{\log(30)}{\log\left(1+\frac{30.}{\text{width}}\right)}}\left(\left(1+\frac{30.}{\text{width}}\right)^{-\log(10)}\left(\frac{x^2}{\text{width}^2}\right)^{\frac{\log(30)}{2}}\right)^{\frac{1}{\log\left(1+\frac{30.}{\text{width}}\right)}}}$$



Window[x] when its slope is changed (below)

center=0;width=50;

$$e^{-25^{-\frac{\log(30)}{\log\left(1+\frac{0.036}{slope}\right)}}\left(\left(1+\frac{0.036}{slope}\right)^{-\log(10)}\left(x^2\right)^{\frac{\log(30)}{2}}\right)^{\frac{1}{\log\left(1+\frac{0.036}{slope}\right)}}}$$